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103

# Exam II , MTH 221 , Fall 2010, 11am section

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**QUESTION 1. (12 points)** Let $F = span\{(1,-1,1),(-1,0,1),(0,-1,2)\}$

(i) Find a basis for $F$.

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$v_3$ is dependent

Basis for $F \Rightarrow B = \{(1,-1,1),(-1,0,1)\}$

(ii) Is $(1,-3,5) \in F$? EXPLAIN.

$$\begin{array}{c} v_1 \\ v_2 \\ w \end{array}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & -3 & 5 \end{bmatrix} \xrightarrow[-R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since we got a zero row in the row that corresponds to W $\Rightarrow$ then that means (w) is dependent $\Rightarrow$ then w can be written as a linear combination of ($V_1$ and $V_2$) which means that **W $\in$ F** which is a span of $V_1$ and $V_2$ ..

**QUESTION 2. (5 points)** Given $v_1 = 1 + x + x^2$, $v_2 = -1 + x - x^2$ are independent in $P_3$. Find $v_3 \in P_3$ such that $B = \{v_1, v_2, v_3\}$ is a basis for $P_3$. Show the work.

$P_3 \longrightarrow R_3$

$V_1: 1+x+x^2 \longrightarrow (1,1,1)$

$V_2: -1+x-x^2 \longrightarrow (-1,1,-1)$

$$\Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ & W & \end{bmatrix} \xrightarrow{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ & W & \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{then } W = (0,0,1) \Rightarrow \boxed{V_3 = x^2} \in P_3 \text{ and}$$

$V_1 = 1+x+x^2$
$V_2 = -1+x-x^2$
$V_3 = x^2$

$\Rightarrow$ are independent $\Rightarrow B = \{1+x+x^2, -1+x-x^2, x^2\}$

**QUESTION 3. (5 points)** Are $(10,5,6),(3,2,8),(-7,45,2),(5,6,23)$ independent in $R^3$?explain (you may finish on the back)

No, they are not independent because they are points in $\mathbb{R}^3$ and the maximum Number of Independent elements in $\mathbb{R}^3$ is (3) points and here we have 4 Points $\Rightarrow$ So, we have at least one point that is a linear combination of the other.

$\Rightarrow$

**QUESTION 4. (15 points)** Given $T : P_3 \longrightarrow R_{2\times 2}$ such that $T(f(x)) = \begin{bmatrix} f(0) & f(1) \\ f(1) & 0 \end{bmatrix}$ is a linear transformation.

(i) Find the standard matrix representation of $T$.

Standard Basis for $P_3$ is $= \{1, x, x^2\}$

$T(1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\Rightarrow$ in $R_4$ they look like $(1,1,1,0)$

$T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad (0,1,1,0)$

$T(x^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad (0,1,1,0)$

So, the standard Matrix representation is $\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ $\boxed{4\times 3}$

Size = dim(codomain) X dim(domain)

$= \boxed{4 \times 3}$

(ii) Find a basis for Ker(T) and write Ker(T) as a span.

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3} \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ stop and read

$x_1 = 0$
$x_2 + x_3 = 0$ $\Rightarrow$ Solution: $x_3 \in \mathbb{R}$, $x_1 = 0$, $x_2 = -x_3$

$N(A) = \{(0, -x_3, x_3) \mid x_3 \in \mathbb{R}\} \Rightarrow \begin{array}{l} x_3 = 1 \\ x_2 = -1 \end{array}$, and $x_1 = 0$

then $N(A) = \text{Span}\{(0,-1,1)\}$

$R^3 \longrightarrow P_3$

$(0,-1,1) \longrightarrow -x + x^2$ $\Rightarrow$ Basis for the $\ker(T) = \{-x + x^2\}$

$\Rightarrow \ker(T) = \text{Span}\{-x + x^2\}$

(iii) Find a basis for Range(T) and write Range(T) as a span.

$\text{Col}(A) = \text{Span}\{(1,1,1,0), (0,1,1,0)\}$

Range(T) lives in $R_{2\times 2}$

$R^3 \longrightarrow R_{2\times 2}$

$\Rightarrow (1,1,1,0) \longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$(0,1,1,0) \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Basis for Range(T) $= \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

$\text{Range}(T) = \text{Span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

**QUESTION 5. (15points)** Given $T : P_3 \longrightarrow R$ is a linear transformation such that $T(2) = 6$, $T(x^2) = -5$, and $T(x^2 + 2x + 1) = 4$.

(i) Find $T(x)$

$T(x^2 + 2x + 1) = T(x^2) + T(2x) + T(1) = T(x^2) + 2T(x)$

$T(2) = 6$

$T(x^2) = -5 \Rightarrow$

$T(x^2 + 2x + 1) = 4$

$\Rightarrow T(x^2) + 2T(x) + T(1) = -5 + 2T(x) + T(1)$

$T(2) = 6 \Rightarrow 2T(1) = 6 \Rightarrow T(1) = 6 \times 1/2 = 3$

$2T(x) - 5 + 3 = 4 \Rightarrow T(x) = (4+5-3)1/2 = 3$

$\boxed{T(x) = 3}$

(ii) Find $T(5x^2 + 7x + 9)$. $= ?$

$5T(x^2) + 7T(x) + 9T(1)$

$= 5(-5) + 7(3) + 9(3) = -25 + 21 + 27 = \boxed{23}$

(iii) Find the standard matrix representation for $T$.

Standard Basis for $P_3 = \{1, x, x^2\}$

$T(1) = 3$

$T(x) = 3 \Rightarrow S.M.R = \underset{1\times 3}{\begin{bmatrix} 3 & 3 & -5 \end{bmatrix}}$

$T(x^2) = -5$

(iv) Find a basis for Ker(T) and write Ker(T) as a span. $\rightarrow [3\ 3\ -5 \mid 0]$

$3x_1 + 3x_2 - 5x_3 = 0$

$x_2, x_3 \in \mathbb{R}$

$x_1 = 1/3 [5x_3 - 3x_2]$

$\boxed{x_1 = 5/3\, x_3 - x_2}$

| $x_2$ | $x_3$ | |
|---|---|---|
| 1 | 0 | $(-1, 1, 0)$ |
| 0 | 1 | $(5/3, 0, 1)$ |

$\Rightarrow N(A) = \text{span}\{(-1,1,0), (5/3, 0, 1)\}$

ker(T) live in $P_3$

Basis for ker(T) $= \{-1 + x, 5/3 + x^2\}$

$N(A) = \{5/3\, x_3 - x_2, x_2, x_3 \mid x_2, x_3 \in R\}$

$\boxed{\ker(T) = \text{Span}\{-1 + x, 5/3 + x^2\}}$

**QUESTION 6. (15 points)** Let $F = \{(a, b, c, d) \in R^4 \mid a, b, c, d \in R, a + 2c + 3d = 0, \text{ and } b - c + d = 0\}$.

(i) Show that $F$ is a subspace of $R^4$.

$a + 2c + 3d = 0$

$b - c + d = 0$

$a = -2c - 3d$

$b = c - d$

$c, d \in R$

$F = \{(-2c - 3d, c - d, c, d) \in R^4 \mid c, d \in R\}$

Since all the co-ordinates of the points are a linear combination of the free variable (c & d) then F is a subspace of $\mathbb{R}^4$

$\Rightarrow -2c - 3d = (-2)c + (-3)d$

$c - d = (1)c + (-1)d$

$c = (1)c + (0)d$

$d = (0)c + (1)d$

(ii) Find a basis for $F$ and write $F$ as a span.

| c | d | |
|---|---|---|
| 1 | 0 | $(-2, 1, 1, 0)$ |
| 0 | 1 | $(-3, -1, 0, 1)$ |

Basis for $F \Rightarrow B = \{(-2, 1, 1, 0), (-3, -1, 0, 1)\}$

$F = \text{Span}\{(-2, 1, 1, 0), (-3, -1, 0, 1)\}$

**QUESTION 7. (8 points)** Let $D = \{3a + (2a + b)x^2 + 4x^3 \mid a, b \in R\}$ Is $D$ a subspace of $P_4$? If NO, explain. If YES, find a basis for $D$

No, D is Not a subspace of $P_4$ because the coefficient of $x^3$ is not a linear combination of the free variables a and b

$\Longleftrightarrow 4 \neq \alpha_1 a + \alpha_2 b \Longleftrightarrow$

**QUESTION 8. (15 points)** Let $A = \begin{array}{c} \begin{array}{cccccc} c_1 & & & c_4 & & c_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 & 2 & 6 \end{bmatrix} \end{array}$

(i) Find a basis for $ROW(A)$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 & 2 & 6 \end{bmatrix} \xrightarrow[-2R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \begin{array}{c} \begin{array}{cccccc} c_1 & & & c_4 & & c_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \end{array}$$

Basis for $Row(A) = \{(1,1,1,1,1,1), (0,0,0,3,4,5), (0,0,0,0,0,4)\}$

(ii) Find a basis for $Col(A)$

Basis for $Col(A) = \{(1,-1,2), (1,2,2), (1,4,6)\}$

**QUESTION 9. (10 points)** Given $L = \{\begin{bmatrix} 3a & 2a+b \\ -b & c \end{bmatrix} \mid a, b, c \in R\}$ is a subspace of $R_{2\times 2}$. Find a basis for $L$.

$a = 1, b = 0, c = 0$ $\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$

$a = 0, b = 1, c = 0$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$a = 0, b = 0, c = 1$ $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Basis for $L = \{\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \\ 0 & \end{bmatrix}$

**QUESTION 10. (BONUS = 3 points)** Given $F = \{(2a + b) + (2a + b)x^3 + (2a + b)x^4 \mid a, b \in R\}$ is a subspace of $P_5$. Find a basis for $F$.

$a = 0,\ b = 1$

$1 + x^3 + x^4$

$a = 1,\ b = 0$

$2 + 2x^3 + 2x^4 \Rightarrow$ But this is a linear Combination of $1 + x^3 + x^4$ and $\alpha = 2$

Basis $= \{1 + x^3 + x^4\}$

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where

$2 + 2x^3 + 2x^4 = \alpha(1 + x^3 + x^4)$

$= 2(1 + x^3 + x^4)$

$= 2 + 2x^3 + 2x^4$